Generating rotating fields in general relativity*

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Abstract. I present a new method to generate rotating solutions of the Einstein–Maxwell equations from static solutions, give several examples of its application, and discuss its general properties.

When dealing with exact stationary solutions of the Einstein equations, one sometimes stumbles on the questions, quite easy to ask, but rather difficult to answer: Given some static solution, what is the family of non-static (rotating) solutions which are near —in some sense— this static solution? On how many parameters do these solutions depend? And (last but not least) how can one practically generate these rotating solutions from the static solution?

In principle, these questions can be answered in the context of the Geroch group [1]. Let us recall that the 4–dimensional stationary Einstein (resp. Einstein–Maxwell) equations are invariant under the group O(2,1) (resp. SU(2,1)) of generalized Ehlers–Harrison transformations [2]. In the case of stationary axisymmetric solutions, with two commuting Killing vectors ∂_t and ∂_φ, the combination of the invariance transformations associated with a given direction in the Killing 2–plane with rotations in this plane leads to the infinite–dimensional Geroch group. These transformations allow in principle the generation of all solutions of the stationary axisymmetric Einstein (or Einstein–Maxwell) problem, which is thus completely integrable. This generation of stationary axisymmetric solutions can be achieved in a

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variety of manners, the most popular being inverse–scattering transform methods applied to the Einstein (or Einstein–Maxwell) problem [3]. While these techniques have led to the construction of many new solutions, until very recently no practical approach to the direct generation of rotating solutions from static solutions was available.

In this talk, I shall exhibit a specific Geroch transformation which achieves this goal in the case of the Einstein–Maxwell theory, using finite combinations of SU(2,1) transformations and global coordinate transformations mixing the two Killing vectors [4, 5]. After recalling briefly the Ernst approach to the reduction of the stationary Einstein–Maxwell problem, I shall describe this direct rotation–generating transformation, and give several examples of its application. I shall then discuss some general properties of the solutions generated by this technique, and mention several promising lines of future research.

In the absence of matter sources, stationary Einstein–Maxwell fields may be parametrized by the metric

$$ds^2 = f\,(dt - \omega_i dx^i)^2 - f^{-1}\,h_{ij}\,dx^i dx^j\,, \tag{1}$$

and the electromagnetic fields

$$F_{i0} = v_{,i}\;, \qquad F^{ij} = f\,h^{-1/2}\epsilon^{ijk}u_{,k}\;, \tag{2}$$

where the fields f, ω_i, v, u and the reduced spatial metric h_{ij} depend only on the space coordinates x^i. The vector potential ω_i may be dualized to the scalar twist potential χ by

$$\chi_i' = -f^2\,h^{-1/2}h_{ij}\,\epsilon^{jkl}\omega_{l,k} + 2(u\,v_{,i} - v\,u_{,i})\,. \tag{3}$$

The complex Ernst potentials are related to the four real scalar potentials f, χ, v and u by

$$\mathcal{E} = f + i\chi - \overline{\psi}\psi\,, \qquad \psi = v + iu\,. \tag{4}$$

The stationary Einstein–Maxwell equations then reduce to the three–dimensional Ernst equations [6]

$$\begin{aligned} f\nabla^2\mathcal{E} &= \nabla\mathcal{E}\cdot(\nabla\mathcal{E} + 2\overline{\psi}\nabla\psi)\,, \\ f\nabla^2\psi &= \nabla\psi\cdot(\nabla\mathcal{E} + 2\overline{\psi}\nabla\psi)\,, \\ f^2R_{ij}(h) &= \mathrm{Re}\left[\frac{1}{2}\mathcal{E}_{,(i}\,\overline{\mathcal{E}}_{,j)} + 2\psi\mathcal{E}_{,(i}\,\overline{\psi}_{,j)} - 2\mathcal{E}\psi_{,(i}\,\overline{\psi}_{,j)}\right]\,, \end{aligned} \tag{5}$$

where the scalar products and covariant Laplacian are computed with the reduced spatial metric h_{ij}, These equations are invariant under an SU(2,1) group of transformations.

The direct rotation–generating transformation is the product

$$\Sigma = \Pi^{-1}\,\mathcal{R}(\Omega,\gamma)\,\Pi \tag{6}$$

of three successive transformations, two "vertical" transformations $\Pi\,,\Pi^{-1}\in$ SU(2,1) acting on the potential space, and a "horizontal" global coordinate transformation $\mathcal{R}(\Omega,\gamma)$ acting on the Killing 2–plane. The transformation Π is the SU(2,1) involution $(\mathcal{E},\psi,h_{ij})\leftrightarrow(\hat{\mathcal{E}},\hat{\psi},\hat{h}_{ij})$ with

$$\hat{\mathcal{E}} = \frac{-1+\mathcal{E}+2\psi}{1-\mathcal{E}+2\psi}\,, \quad \hat{\psi} = \frac{1+\mathcal{E}}{1-\mathcal{E}+2\psi}\,, \quad \hat{h}_{ij} = h_{ij}\,. \tag{7}$$

The resulting transformation of the gravitational potential f is $\hat{f} = 4f/|1-\mathcal{E}+2\psi|^2$. Starting from the Schwarzschild solution, written in prolate spheroidal coordinates [7] as

$$\begin{aligned} ds^2 &= f dt^2 - f^{-1}m^2\,[dx^2 + (x^2-1)(d\theta^2+\sin^2\theta\, d\varphi^2)]\,, \\ \mathcal{E} &= f = \frac{x-1}{x+1}\,, \qquad \psi = 0 \end{aligned} \tag{8}$$

(the coordinate x is related to the "standard" radial coordinate r by $x = (r-m)/m$), the action of Π leads, after rescaling the time coordinate to $\tau = m^{-1}t$, and putting $y = \cos\theta$, to the open Bertotti–Robinson (BR) solution [8]

$$\begin{aligned} d\hat{s}^2 &= m^2\left[(x^2-1)\,d\tau^2 - \frac{dx^2}{x^2-1} - \frac{dy^2}{1-y^2} - (1-y^2)\,d\varphi^2\right]\,, \\ \hat{\mathcal{E}} &= -1\,, \qquad \hat{\psi} = x\,. \end{aligned} \tag{9}$$

This non–asymptotically flat spacetime is the direct product $\mathrm{adS}_2\times\mathrm{S}^2$ of two constant curvature two–dimensional spaces. More generally, if the initial Ernst potentials have the asymptotic ($r\to\infty$) monopole behaviour $(\mathcal{E},\psi)\to(\mathcal{E}_\infty,\psi_\infty)+O(1,r)$, with

$$\mathcal{E}_\infty = 1 + 2\psi_\infty\,, \tag{10}$$

the transformation Π leads to asymptotically BR–like potentials $(\hat{\mathcal{E}},\hat{\psi})\to$ (const., $O(r)$) with $f\to O(r^2)$.

The global coordinate transformation $\mathcal{R}(\Omega,\gamma)$ is the product of the transformation to a uniformly rotating frame and of a time dilation,

$$\begin{aligned} d\varphi &= d\varphi' + \Omega\gamma\, dt' \,, \\ dt &= \gamma\, dt' \,. \end{aligned} \tag{11}$$

In the case of electrostatic solutions with $\hat{\mathcal{E}}$ and $\hat{\psi}$ real ($\hat{\omega}=0$), the frame rotation gives rise to an induced gravimagnetic field $\hat{\omega}'$ as well as to an induced magnetic field. However this transformation does not modify the leading asymptotic behavior of the BR metric or of asymptotically BR–like metrics. Because of this last property, the last transformation Π^{-1} in (6) then leads to asymptotically flat, but complex, Ernst potentials corresponding to a monopole–dipole solution.

For instance, the transformation (11) leads from the BR solution (9) to

$$\begin{aligned} d\hat{s}'^2 &= m^2 \,[\, \gamma^2 (x^2+\eta^2 y^2 - (1+\eta^2))(d\tau' - \frac{\eta(1-y^2)}{\gamma(x^2+\eta^2y^2-(1+\eta^2))}\, d\varphi')^2 \\ &\quad - \frac{dx^2}{x^2-1} - \frac{dy^2}{1-y^2} - \frac{(x^2-1)(1-y^2)}{x^2+\eta^2y^2-(1+\eta^2)}\, d\varphi'^2 \,] \,, \\ \hat{\mathcal{E}}' &= -\gamma^2(1+\eta^2) \,, \qquad \hat{\psi}' = \gamma(x - i\eta y) \,, \end{aligned} \tag{12}$$

with $\eta \equiv m\Omega$. The Ernst potentials $\hat{\mathcal{E}}'$,$\hat{\psi}'$ describe the same static BR fields as the Ernst potentials $\hat{\mathcal{E}}$,$\hat{\psi}$ of the original BR solution, but in a different coordinate frame. For the special choice $\gamma = (1+\eta^2)^{-1/2}$ (corresponding to $\hat{\mathcal{E}}' = -1$), the final transformation Π^{-1} then leads to the Ernst potentials

$$\mathcal{E}' = \frac{px - iqy - 1}{px - iqy + 1} \,, \qquad \psi' = 0 \tag{13}$$

(with $p = (1+\eta^2)^{-1/2}$, $q = \eta(1+\eta^2)^{-1/2}$), which are those of the Kerr solution [4]. Summarizing, the Kerr solution has been generated from the Schwarzschild solution by the transformation $\Sigma = \Pi^{-1}\mathcal{R}\Pi$. The choice of another value for γ would lead after the second step to a constant potential $\hat{\mathcal{E}}' \neq -1$, resulting in a final electromagnetic potential $\psi' \neq 0$. So for a generic value of γ the transformation $\Sigma = \Pi^{-1}\mathcal{R}\Pi$ generates the Kerr–Newman family of solutions from the Schwarzschild solution.

Another example is that of the Voorhees–Zipoy family of static vacuum solutions [7], depending on a real parameter δ, with the Ernst potentials (in prolate spheroidal coordinates)

$$\mathcal{E} = \left(\frac{x-1}{x+1}\right)^{\delta} \,, \qquad \psi = 0 \,. \tag{14}$$

Carrying out the spin–generating transformation Σ, with γ chosen such that the resulting solutions are electrically neutral, we have obtained in [4] a family of rotating solutions depending continuously on three parameters. These solutions, which have a dipole magnetic moment and a quadrupole electric moment, are clearly different from the discrete Tomimatsu–Sato [9] family of vacuum rotating solutions, which also reduce to Voorhees–Zipoy solutions in the static limit, but are known only for integer δ.

Yet another example is the construction of self–gravitating cosmic rings. It has been shown [10] that superconducting closed cosmic strings, which would otherwise collapse under the effect of their own tension, can be stabilized by rotation. As shown in [5], such rotating cosmic strings or vortons without thickness can be obtained as self–consistent solutions to the Einstein–Maxwell equations. The starting point is the charged static "ring wormhole" solution [11], given in oblate spheroidal coordinates (x, y) by

$$\mathcal{E} = +1\,, \quad \psi = \cot\sigma\,, \qquad \sigma(x) = \sigma_0 + \alpha \arctan x\,. \tag{15}$$

The corresponding spacetimes have a circular topological defect (cosmic ring) at $x = y = 0$. The transformation Σ generates from these static solutions a family of exact rotating solutions depending on 4 parameters, which reduce to 3 in the electrically neutral case; these parameters are further constrained by the condition that the ring $x = y = 0$ rotates at subluminal velocities. The resulting spacetimes have an "extra" point at spatial infinity (which however cannot be reached by timelike or null geodesics) and may also present (depending on the parameters) an extra ring singularity.

In the case of a generic axisymmetric electrostatic solution ($\mathcal{E}$, ψ real), it is convenient to choose Weyl coordinates ρ, z, φ such that

$$\omega_i\,dx^i \equiv \omega(\rho, z)\,d\varphi\,, \qquad h_{ij}\,dx^i\,dx^j = \mathrm{e}^{2k(\rho,z)}(d\rho^2 + dz^2) + \rho^2\,d\varphi^2\,. \tag{16}$$

The first two Ernst equations (5) for the transformed BR–like Ernst potentials $\hat{\mathcal{E}}$, $\hat{\psi}$ then take the real form

$$\nabla(\rho \hat{f}^{-1}\nabla\hat{\mathcal{E}}) = 0\,, \qquad \nabla(\rho \hat{f}^{-1}\nabla\hat{\psi}) = 0\,, \tag{17}$$

which imply the existence of dual Ernst potentials $\hat{\mathcal{F}}$, $\hat{\phi}$ such that

$$\hat{\mathcal{F}}_{,m} = \rho \hat{f}^{-1}\,\epsilon_{mn}\hat{\mathcal{E}}_{,n}\,, \quad \hat{\phi}_{,m} = \rho \hat{f}^{-1}\,\epsilon_{mn}\hat{\psi}_{,n} \tag{18}$$

(m,$n = 1$,2, with $x^1 = \rho$, $x^2 = z$). It may then be shown that the transformation $\mathcal{R}$ with $\gamma = 1$ transforms the potentials ($\hat{\mathcal{E}}$, $\hat{\psi}$, $\mathrm{e}^{2\hat{k}}$) into

$$\begin{aligned}
\hat{\mathcal{E}}' &= \hat{\mathcal{E}} + 2i\Omega\,(z + \hat{\mathcal{F}} + \hat{\psi}\hat{\phi}) - \Omega^2(\rho^2/\hat{f} + \hat{\phi}^2)\,, \\
\hat{\psi}' &= \hat{\psi} + i\Omega\hat{\phi}\,, \qquad \mathrm{e}^{2\hat{k}'} = (1 - \Omega^2\rho^2/\hat{f}^2)\,\mathrm{e}^{2\hat{k}}\,.
\end{aligned} \tag{19}$$

From these one may write down the asymptotically flat potentials $\mathcal{E}'$, ψ', from which the rotating metric $g'_{\mu\nu}$ and the rotating electromagnetic potentials A'_μ may be obtained by solving duality equations. A partial solution of this problem is (again for $\gamma = 1$):

$$\begin{aligned} f' &= (|\hat{F}|^2/|\hat{F}'|^2)\lambda\, f\,, \qquad \mathrm{e}^{2k'} = \lambda\, \mathrm{e}^{2k}\,, \\ \omega'{}_{,m} &= \Omega^{-1}|\hat{F}'|^2(\lambda^{-1})_{,m} - (2\rho/\hat{f})\lambda^{-1}\epsilon_{mn}\,\mathrm{Im}(\bar{\hat{F}}'\,\hat{F}'{}_{,n})\,, \end{aligned} \tag{20}$$

with

$$|\hat{F}'| \equiv |1 - \hat{\mathcal{E}}' + 2\hat{\psi}'|/2 = 1/|F'|\,, \quad \lambda \equiv (1 - \Omega^2\rho^2/\hat{f}^2)\,. \tag{21}$$

From (20) one can derive the main generic properties of these rotating Einstein–Maxwell solutions:

1) They are *regular* on the axis $\rho = 0$,

$$\mathrm{e}^{2k'} = 1\,, \qquad \partial_z\omega' = 0\,, \tag{22}$$

if the original static fields are regular.

2) They present *ergospheres* $f'(\rho, z) = 0$ for $\hat{f}(\rho, z) = \pm\Omega\rho$. Near such zeroes of f', the rotating metric

$$ds'^2 \simeq \mp 2\rho\, dt\, d\varphi \mp (\hat{F}'^2/\Omega\rho)(\mathrm{e}^{2k}(d\rho^2 + dz^2) + \rho^2\, d\varphi^2) \tag{23}$$

is non–degenerate.

3) The rotating solutions present *horizons* for

$$f'\mathrm{e}^{-2k'} = \frac{|F'|^2}{|F|^2}\, f\, \mathrm{e}^{-2k} = 0\,. \tag{24}$$

These may occur for either

–a) zeroes of f (horizons of the static solution); an illustration of this horizon conservation is the fact that the transformation Σ (with arbitrary Ω and γ) transforms an extreme Reissner–Nordström black hole ($M^2 = Q^2$) into an extreme Kerr–Newman black hole ($M'^2 = Q'^2 + a'^2$); or

–b) poles of F (leading to simple zeroes of $f/|F|^2$); an example is the inner horizon of the Kerr–Newman solution, which is generated from the singularity $r = 0$ of the Schwarzschild solution.

4) The rotating metric (20) may present Kerr–like *ring singularities* corresponding to the zeroes of the function $|\hat{F}'|^2(\rho, z)$. In the plane–symmetric case, these rings are located in the plane $z = 0$ ($\mathrm{Im}\hat{F}' = 0$) with radii given by the solutions of the equation

$$2\,\mathrm{Re}\,\hat{F}' = (1 + \hat{\psi})^2 - \hat{f}' = 0 \qquad \text{for } z = 0 \tag{25}$$

$(\hat{f}' \equiv \hat{f} - \Omega^2\rho^2/\hat{f})$. Near such rings the rotating metric reduces to

$$ds'^2 \equiv \frac{\hat{f}'_0}{|\hat{F}'|^2}\,(dt - \omega'_0\,d\varphi)^2 - \frac{|\hat{F}'|^2}{\hat{f}'_0}\,(\mathrm{e}^{2\hat{k}'_0}(d\rho^2 + dz^2) + \rho_0^2\,d\varphi^2)\,, \qquad (26)$$

with $\hat{f}'_0 \geq 0$ from Eq. (25), so that the rotating ring singularity is timelike if $\hat{f}'_0 \neq 0$ (the Kerr ring corresponds to the very special case in which $\hat{f}'_0$ also vanishes on the ring). In the case of the solutions studied in [5], these unwanted ring singularities may be avoided by suitably choosing the parameters of the static solution and/or the parameter Ω.

There are surely other interesting applications of this solution–generating technique besides those mentioned here. At a more fundamental level, there remain a number of open questions.

A first question is that of the generality of the transformation Σ. For instance, how many inequivalent rotating solutions can be generated from a given static solution by transformations $\mathcal{U}\Sigma\mathcal{U}^{-1}$, with $\mathcal{U} \in \mathrm{SU}(2,1)$? For those transformations $\mathcal{U}$ which conserve staticity, the resulting solutions are found to belong the same family (with different parameter values), however the case of more general transformations $\mathcal{U}$ has not yet been elucidated. A related question is that of the precise connection of the direct spin–generating method presented here with other spin–generating techniques. This method could also in principle be generalized to other gravitating field theories with a highly symmetric stationary target space. Two cases are presently under investigation, those of five–dimensional Kaluza–Klein theory, and of dilaton–axion gravity with one Abelian gauge field.

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